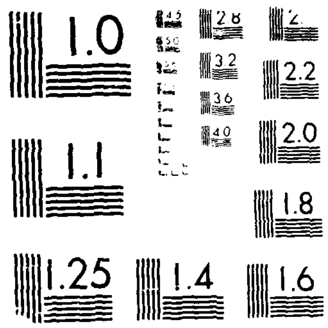
AD-A192 908   INVERSE PROBLEMS AND A UNIFIED APPROACH TO   1/1
INTEGRABILITY IN 1 1+1 AND 2+1 DIMENSIONS(U) CLARKSON
UNIV POTSDAM NY DEPT OF MATHEMATICS AND COMPUTER SCI..
UNCLASSIFIED   A S FOKAS ET AL. 1986 N00014-86-K-0603   F/G 12/2   NL



MICROCOPY RESOLUTION TEST CHART

INVERSE PROBLEMS AND A UNIFIED APPROACH TO INTEGRABILITY IN 1, 1+1 AND 2+1 DIMENSIONS*

A. S. Fokas and V. Papageorgiou
Department of Mathematics and Computer Science
and Institute for Nonlinear Studies
Clarkson University, Potsdam, New York 13676

ABSTRACT

N00014-86-K-0603
1986

1. INTRODUCTION

The aim of this paper ~~is to~~ emphasize that there exists a unified approach for solving initial value problems for equations in 1, 1+1 (i.e., one spatial and one temporal), and 2+1 (i.e., two spatial and one temporal) dimensions. Furthermore it remarks on inverse problems in higher than two spatial dimensions. Although these inverse problems are not related to physically significant nonlinear evolution equations, they are useful in the context of inverse scattering. In this presentation we emphasize the main ideas. The detail formalisms can be found in the cited papers.

It turns out that solving the initial value problem for some equations for $q(t)$, or $q(x,t)$, or $q(x,y,t)$ is equivalent to solving an inverse problem for some related eigenfunction $\Psi(z;t)$, or $\Psi(z;x,t)$, or $\Psi(z;x,y,t)$. The inverse problem takes the form of a Riemann-Hilbert (RH) problem for equations in 1 and 1+1, and the form of a nonlocal RH problem or of a $\bar{\partial}$(DBAR) problem for equations in 2+1 (a DBAR problem is generalization of a RH problem). To define the relevant RH or DBAR

*This article consists of expanded material of six lectures presented by one of us (A. S. Fokas) at the Workshop on "Solitons," Tiruchirapalli, India, January 1987.

problems one needs to study the analyticity properties of Ψ with respect to z. Furthermore those problems are uniquely defined in terms of certain asymptotic data of the underlying linear system satisfied by Ψ: **Monodromy data** in 1, **scattering data** in 1+1 and some cases of 2+1, and **inverse data** in some cases of 2+1. We use the Painlevé IV(PIV), modified KdV (mKdV) and the Davey-Stewartson (DS) as illustrative examples of equations in 1, 1+1, and 2+1 respectively.

The above inverse problems can be naturally generalized to higher than two spatial dimensions. For example, the generalization of the inverse problem associated with the DS equation leads to an inverse problem for a matrix valued function $\Psi(z;x_0,x)$, $z \in \mathbb{C}^n$, $x_0 \in \mathbb{R}^1$, $x \in \mathbb{R}^n$, $n > 1$. However, while the associated potential $q(x_0,x)$ depends on n+1 variables, the inverse data $T(z_R,z_1,m_2,\ldots,m_n)$, $z_R \in \mathbb{R}^n$, $z_1 \in \mathbb{R}^n$, $m_\ell \in \mathbb{R}$, depends on 3n-1 variables. This has important implications: (a) The inverse data must be appropriately constrained. This "characterization" of the inverse data is conceptually analogous to the characterization of the inverse scattering data in the multidimensional Schrödinger equation [1]. (b) The existence of "redundant" scattering parameters can be used to reduce the above problem to one in two spatial dimensions. This is in contrast to the case of the multidimensional Schrödinger equation where the inverse problem can be solved in closed form. (c) Since the inverse problem for Ψ is reduced to one in two spatial dimensions, it follows that, if one allows Ψ, q to depend parametrically on t, $q(x_0,x,t)$ satisfies an evolution equation reducible to two spatial dimensions. In particular, the N-wave interaction equation in n+1 spatial dimensions can always be reduced to two spatial dimensions. Thus a genuine three-spatial-dimensional nonlinear evolution equation, related to an inverse problem, remains to be found. (It should be noted that several other "multidimensional" problems can be reduced to one or two spatial dimensions, see M. J. Ablowitz's contribution in these proceedings.)

We first define the standard RH and DBAR problems.

2. RH AND DBAR PROBLEMS

Let C be a simple, smooth closed contour dividing the complex z-plane into two regions D^+ and D^- (the positive direction of C will be taken as that for which D^+ is on the left).

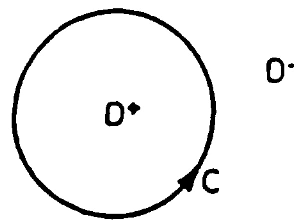


Fig. 1

A function $\phi(z)$ defined in the entire plane, except for points on C, will be called sectionally holomorphic if: (i) the function $\Phi(z)$ is holomorphic in each of regions D^+ and D^- except, perhaps, at $z = \infty$; (ii) the function $\Phi(z)$ is sectionally continuous with respect to C, approaching the definite limiting values $\Phi^+(\zeta)$, $\Phi^-(\zeta)$ as z approaches a point ζ on C from D^+, or D^-, respectively. The classical homogeneous RH problem is defined as follows [2]. Given a contour C, and a function $G(\zeta)$ which is Hölder on C and $\det G(\zeta) \neq 0$ on C, find a sectionally holomorphic function $\Phi(z)$, with finite degree at ∞, such that

$$\Phi^+(\zeta) = G(\zeta)\Phi^-(\zeta), \quad \text{on } C, \tag{2.1}$$

where $\Phi^\pm(\zeta)$ are the boundary values of $\Phi(z)$ on C. If $G(\zeta)$ is scalar, (2.1) is solvable in closed form. If $G(\zeta)$ is a matrix valued function, then (2.1) is in general solvable in terms of a system of Fredholm integral equations. Various generalizations of the above RH problem are possible. For example: (i) The contour C may be replaced by a union of intersecting contours. (ii) $G(\zeta)$ may have simple discontinuities at a finite number of points; in this case one allows $\Phi(z)$ to have integrable singularities in the neighbourhood of these points. (iii) RH problems may be considered in other than Hölder spaces (e.g. [3]): (iv) One may consider inhomogeneous RH problems $\Phi^+(\zeta) = G(\zeta)\Phi^-(\zeta) + F(\zeta)$ on C.

The DBAR problem can be defined as follows: Given $\partial\Phi/\partial\bar{z}$, find Φ. If $\partial\Phi/\partial\bar{z} = 0$ everywhere except on a curve, then the DBAR problem reduces to a RH problem (since $\partial\Phi/\partial\bar{z} = \Phi^+ - \Phi^-$, in a distribution sense). The DBAR problem can be solved via the following extension of Cauchy's

formula [4]

$$\Psi(z) = \frac{1}{2\pi i}\int_{\mathbb{R}^2} d\bar{\zeta}\wedge d\zeta \, \frac{1}{\zeta - z}\frac{\partial\Psi(\zeta)}{\partial\bar{\zeta}} + \frac{1}{2\pi i}\int_C d\zeta \, \frac{\Psi(\zeta)}{\zeta - z}. \tag{2.2}$$

It is interesting that the first RH problem was formulated in connection with an inverse problem (see [12] for references). Actually RH problems are intimately related to solutions of inverse problems in 1+1, 2+1, and 1 dimensions:

3. INVERSE PROBLEMS IN 1+1

We recall that a necessary condition for a given nonlinear equation for $q(x,t)$ to be solvable via IST is that this equation is the compatibility condition of a Lax pair of linear equations. Let us consider the modified KdV equation

$$q_t + q_{xxx} - 6q^2 q_x = 0, \tag{3.1}$$

as an illustrative example. Equation (3.1) is the compatibility condition of

$$\Psi_x(z;x,t) = iz[J\ ,\ \Psi(z;x,t)] + Q\Psi(z;x,t);$$

$$J \doteq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q \doteq \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} \tag{3.2a}$$

$$\Psi_t(z;x,t) = [U_0, \Psi(z;x,t)] + \tilde{Q}\Psi(z;x,t) \tag{3.2b}$$

$$U_0 = \begin{pmatrix} -4iz^3 & 0 \\ 0 & 4iz^3 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} -2izq^2 & 4qz^2 + 2iq_x z + 2q^3 - q_{xx} \\ 4qz^2 - 2iq_x z + 2q^3 - q_{xx} & 2iq^2 z \end{pmatrix}$$

We first note that the above Lax pair is isospectral, i.e., $\frac{dz}{dt} = 0$. Also it turns out that equation (3.2a) is of primary importance; equation (3.2b) plays only an auxiliary role. To solve the initial value problem for initial data decaying as $|x| \to \infty$, one first formulates an inverse problem for $\Psi(z;x,t)$: Given appropriate scattering data reconstruct Ψ.

By studying the analytic properties of Ψ with respect to z, where Ψ satisfies (3.2a) one establishes the existence of a Ψ which is a sectionally meromorphic function of z, with a jump along the Re z axis. This jump as well as the residues of the poles, are given in terms of appropriate scattering data. **Thus the inverse problem is equivalent to a matrix, regular, continuous RH problem defined along the Re z axis and uniquely specified in terms of scattering data.**

Since in the above discussion we have only used (3.2a), it is evident that one may pose an inverse problem for any function q(x). However, this result is useful for solving the initial value problem for q(x,t) only if q evolves in such a way in t, that the scattering data is known for all t. If Ψ evolves in t according to (3.2b) (i.e., if q solves (3.1)) then it turns out that the evolution of the scattering data with respect to t is simple. Hence, the above RH problem is specified in terms of initial scattering data; its solution yields $\Psi(z;x,t)$ and then (3.2a) gives q(x,t).

We summarize the results of [5],[13] concerning mKdV in the case of solitonless potentials.

Proposition 3.1 (Bounded eigenfunctions). A solution of (3.2a) bounded for all complex values of $z = z_R + iz_I$ and tending to I as $z \to \infty$ is given by

$$\Psi(z;x) = \begin{cases} \Psi^+(z;x), & z_I > 0 \\ \Psi^-(z;x), & z_I < 0 \end{cases} \qquad (3.3)$$

where $\Psi^\pm(z;x)$ satisfy the following integral equations:

$$\Psi^\pm(z;x) = I + \int_{-\infty}^{x} d\xi e^{iz(x-\xi)\hat{J}} \pi_\pm Q(\xi)\Psi^\pm(z;\xi) - \int_{x}^{+\infty} d\xi\, e^{iz(x-\xi)\hat{J}}(\pi_\mp + \pi_0)Q\Psi^\pm \qquad (3.4)$$

where if F and Y are 2 × 2 matrices then

$$e^{\hat{Y}}F = e^{Y}Fe^{-Y}, \quad \pi_{+}F \doteq \begin{pmatrix} 0 & F_{12} \\ 0 & 0 \end{pmatrix}, \ \pi_{-}F = \begin{pmatrix} 0 & 0 \\ F_{21} & 0 \end{pmatrix} \quad (3.5)$$

$$\pi_0 F = \mathrm{Diag}(F_{11}, F_{22}).$$

Proposition 3.2 (Departure from Holomorphicity-Scattering Equation). Ψ^{+}, Ψ^{-} are holomorphic functions of z for $z_I > 0$, $z_I < 0$ respectively. The departure from holomorphicity for $z = z_R$ is given by

$$\Psi^{+}(z;x) - \Psi^{-}(z;x) = \Psi^{+}e^{izx\hat{J}}(I - B^{-1}(z)b(z)) \quad (3.6)$$

where

$$B(z) \doteq I + \int_{-\infty}^{\infty} d\xi e^{-iz\xi\hat{J}}\pi_{+}(Q\Psi^{+}), \quad b(z) \doteq I + \int_{-\infty}^{\infty} d\xi e^{-iz\xi\hat{J}}\pi_{-}(Q\Psi^{-})$$

so,

$$\Psi^{+}(z;x)e^{iz\xi\hat{J}}(B^{-1}(z)b(z)) = \Psi^{-}(z;x). \quad (3.7)$$

Proposition 3.3 (Inverse Problem-Reconstruction of Q)

Q(x) is obtained from

$$Q(x) = [J, \frac{1}{2\pi}\int_{-\infty}^{\infty} dz'\Psi(z';x)e^{iz'x\hat{J}}(I - B^{-1}(z')b(z'))] \quad (3.8)$$

where $\Psi(z;x)$ solves the following Riemann-Hilbert boundary value problem:

$$\Psi(z;x) = I + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{dz'\Psi(z';x)e^{iz'x\hat{J}}(I - B^{-1}(z')b(z'))}{z' - (z - i0)} \quad (3.9)$$

Using equation (3.2b) we obtain:

Proposition 3.4 (Evolution of Scattering Data). The evolution of the scattering data from $B(z;0)$, $b(z;0)$ is given by

$$B(z;t) = e^{\hat{U}_0 t}B(z;0), \quad b(z;t) = e^{\hat{U}_0 t}b(z;0).$$

Since B (resp. b) is a strictly upper (resp. lower) triangular matrix the evolution of the scattering data is given by

$$B_{12}(z;t) = e^{-8iz^3t}B_{12}(z;0), \quad b_{21}(z;t) = e^{8iz^3t}b_{21}(z;0). \qquad (3.10)$$

4. INVERSE PROBLEMS IN 2+1

Let us consider the Davey-Stewartson equation (a two dimensional analogue of the nonlinear Schrödinger equation)

$$iQ_t + \frac{1}{2}(\sigma^2 Q_{xx}+Q_{yy}) = -\sigma^2\lambda|Q|^2Q + \Phi Q, \quad \Phi_{xx} - \sigma^2\Phi_{yy} = 2\lambda\sigma^2(|Q|^2)_{xx};$$

$$\lambda = \pm 1 \qquad (4.1)$$

as an illustrative example. A Lax pair for (4.1) is given by

$$\Psi_x = iz(J\Psi-\Psi J) + q\Psi + \sigma J\Psi_y, \quad J \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q \doteq \begin{pmatrix} 0 & Q \\ \bar{Q}\lambda & 0 \end{pmatrix} \qquad (4.2a)$$

$$\Psi_t = A_3\Psi_{yy} + A_2\Psi_y + A_1\Psi - z^2(A_3\Psi-\Psi A_{30}) + 2izA_3\Psi_y + izA_2\Psi, \qquad (4.2b)$$

where A_1, A_2, A_3, A_{30} are appropriate matrix functions of Q, $\bar{Q}$ (The bar denotes complex conjugate).

The situation is conceptually similar to the case of 1+1: To solve the initial value problem for $q(x,y,t)$ one first formulates an inverse problem for $\Psi(z;x,y,t)$. Depending on the value of σ there exist two different cases (for brevity of presentation we assume non-existence of poles, i.e., non-existence of lumps): (i) $\sigma = 1$. There exists a Ψ which is a sectionally holomorphic function of z and which has a jump along the Re z axis. This jump is also given in terms of scattering data but it depends on them in a non-local way. Thus the inverse problem is equivalent to a non-local, matrix continuous RH problem defined along the Re z axis and uniquely specified in terms of scattering data. (ii) $\sigma = -i$. There exists a Ψ which is bounded for all complex z, but which is analytic nowhere in the complex z plane. However, its departure from holomorphicity $\partial\Psi/\partial\bar{z}$ can be expressed in terms of appropriate inverse data. Thus, now the inverse problem is equivalent to a $\bar{\partial}$ (DBAR) problem: Given $\partial\Psi/\partial\bar{z}$ reconstruct Ψ.

Using (4.2b), again one shows that the inverse scattering and the inverse data evolve simply in time. Hence, the above RH and $\bar{\partial}$ problems are specified in terms of initial data; their solutions yield $\Psi(z;x,y,t)$

and then (4.2a) gives $q(x,y,t)$.

We summarize the results of [6] concerning DSI ($\sigma = 1$, Proposition 4.1.-4.4) and DSII ($\sigma = -i$, Propositions 4.5-4.8).

Proposition 4.1 (Bounded Eigenfunctions) A solution of (4.2a) with $\sigma = 1$ bounded for all complex values of $z = z_R + iz_I$ and tending to I as $z \to \infty$ is given by

$$\Psi(z;x,y) = \begin{cases} \Psi^+(z;x,y), & z_I > 0 \\ \Psi^-(z;x,y), & z_I < 0 \end{cases} \tag{4.3}$$

where $\Psi^{\pm}(z;x,y)$ satisfy the following integral equations:

$$\Psi^{\pm}(z;x,y) = I + \frac{1}{2\pi}\int_{-\infty}^{x} d\xi\, e^{iz(x-\xi)\hat{J}}\int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} dm\, e^{im[(y-\eta)I+(x-\xi)J]}\cdot (\pi_0+\pi_{\pm})(q(\xi,\eta)\Psi^{\pm}(z;\xi,\eta))$$

$$- \frac{1}{2\pi}\int_{x}^{\infty} d\xi\, e^{iz(x-\xi)\hat{J}}\int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} dm\, e^{im[(y-\eta)I + (x-\xi)J]}\cdot \pi_{\mp}(q(\xi,\eta)\Psi^{\pm}(z;\xi,\eta)) \tag{4.4$^\pm$}$$

(cf. (3.5) for notation).

Assuming that the linear integral equations (4.4)$^{\pm}$ have no homogeneous solutions, it follows that:

Proposition 4.2 (Departure from Holomorphicity). Ψ^+, Ψ^- are holomorphic functions of z for $z_I > 0$, $z_I < 0$, respectively. Hence the function $\Psi(z;x,y)$ defined by (4.3) is a sectionally holomorphic function of z. In particular, $\frac{\partial\Psi}{\partial\bar{z}} = 0$ for all z, with $z_I \neq 0$ and $\frac{\partial\Psi}{\partial\bar{z}} = \Psi^+(z;x,y) - \Psi^-(z;x,y)$ for $z = z_R$. The departure from holomorphicity is given by:

$$\Psi^+(z;x,y) - \Psi^-(z;x,y) = \int_{-\infty}^{\infty} dz'\,\Psi^-(z';x,y)e^{iz'Jx+iz'y} f(z',z)e^{-izJx-izy}, \tag{4.5}$$

for $z = z_R$, where the scattering data $f(z',z)$ are given by:

$$f_{11}(z',z) = -\int_{-\infty}^{\infty} dm\ f_{12}(z',m)f_{21}(m,z),\ f_{12}(z',z)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\ Q\Psi_{22}^{-}\ e^{-(z+z')\xi+i(z-z')\eta}$$

$$f_{21}(z',z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\lambda\ \tilde{Q}\Psi_{11}^{+}\ e^{i(z+z')\xi+i(z-\ell)\eta},\ f_{22}=0. \quad (4.7)$$

Proposition 4.3 (Inverse Problem-Reconstruction of the Potential q) $q(x,y)$ is obtained from:

$$q(x,y) = -\frac{1}{2\pi}[J, \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dz\Psi\ (z';x,y)e^{iz'Jx}f(z',z)e^{-izJx+i(z'-z)y}] \quad (4.8)$$

where $\Psi^{-}(z;x,y)$ solves the following integral equation:

$$\Psi\ (z;x,y) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz'' \int_{-\infty}^{\infty} dz'\ \frac{\Psi\ (z'';x,y)e^{iz''Jx}f(z'',z')e^{-iz'Jx+i(z''-z')y}}{z' - z + i0}$$

$$= I. \quad (4.9)$$

Finally from (4.2b) we obtain the following:

Proposition 4.4 (Evolution of the Scattering Data). The evolution of the scattering data from $t = 0$, $f(z',z;0)$ is given by:

$$f(z',z;t) = e^{-z'^{2}tA_{30}}\ f(z',z;0)e^{z^{2}tA_{30}} \quad (4.10)$$

where

$f(z',z;0)$ is given by (4.7) and $A_{30} = \mathrm{diag}(i,-i)$.

Proposition 4.5 (Bounded Eigenfunctions). A solution of (4.2a) with $\sigma = -i$ bounded for all complex values of $z = z_R + iz_I$ and tending to I as $z \to \infty$ satisfies the following Fredholm linear integral equation

$$\Psi(z;x,y) = I + (G_{z_R,z_I,q}\Psi(z;.,.))(x,y) \quad (4.11)$$

where

$$((G_{z_R,z_I,q}\Psi(z;.,.)))_{1j} + \frac{1}{2\pi}(\int_{-\infty}^{x} d\xi\int_{-\infty}^{c_{ij}z_I} dm\int_{-\infty}^{\infty} d\eta - \int_{x}^{\infty} d\xi\int_{c_{ij}z_I}^{\infty} dm\int_{-\infty}^{\infty} d\eta) \times$$

$$\{\exp[\ (m+i(1-J_j)z)(x-\xi)+im(y-\eta)][q(\xi,\eta)\Psi(z;\xi,\eta)]\}_{ij}, \quad (4.12)_{ij}$$

and

$$((G_{z_R z_I}, q\Psi(z;\cdot,\cdot))_{2j} \doteq \frac{1}{2\pi}\Big(\int_{-\infty}^{x} d\xi \int_{c_{2j}z_I}^{\infty} dm \int_{-\infty}^{\infty} d\eta - \int_{x}^{\infty} d\xi \int_{-\infty}^{c_{2j}z_I} dm \int_{-\infty}^{\infty} d\eta\Big) \times$$

$$\{\exp[-(m+i(1+J_j)z)(x-\xi)+im(y-\eta)][q(\xi,\eta)\Psi(z;\xi,\eta)]\}_{2j}, \qquad (4.12)_{2j}$$

$c_{1j} = 1-J_j$, $c_{2j} = 1+J_j$, $j = 1,2$.

Proposition 4.6 (Departure from Holomorphicity). For every $z \in \mathbb{C}$

$$\frac{\partial \Psi}{\partial \bar{z}}(z;x,y) = \Psi(\bar{z};x,y)\Omega(z_R,z_I;x,y) \qquad (4.13)$$

where the matrix Ω is defined by: $\Omega_{11} = \Omega_{22} = 0$

$$\Omega_{ij} \doteq T_{ij}(z)\exp\theta_{ij}(z;x,y),\ i \neq j \qquad (4.14)$$

$$T_{ij}(z) \doteq \frac{i}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \{q(\xi,\eta)\Psi(z;\xi,\eta)\}_{ij} \exp\{-\theta_{ij}(z;\xi,\eta)\},\ i \neq j$$

$$\theta_{12}(z;x,y) \doteq 2i(xz_R + yz_I),\ \theta_{21}(z;x,y) = 2i(-xz_R+yz_I).$$

Propostion 4.7 (Inverse Problem-Reconstruction of q). $q(x,y)$ is obtained from

$$q(x,y) = [J, \frac{1}{2\pi}\iint_{\mathbb{C}} \Psi(\bar{z};x,y)\ \Omega(z_R,z_I;x,y)dz \wedge d\bar{z}] \qquad (4.15)$$

where $\Psi(z;x,y)$ satisfies:

$$\Psi(z;x,y) - \frac{1}{2\pi i}\iint_{\mathbb{C}} \Psi(\bar{z};x,y)\ \Omega(z_R',z_I';x,y)\ \frac{dz' \wedge d\bar{z}'}{z'-z} = I. \qquad (4.16)$$

Finally from equation (4.2b) we obtain:

Proposition 4.8 (Evolution of the Inverse Data). The inverse data at time t, $\Omega(z_R,z_I;x,y,t)$, is given by

$$\Omega(z_R,z_I;x,y,t) = \exp(\bar{z}^2 A_{30}t)\ \Omega(z_R,z_I;x,y,0)\exp(-z^2 A_{30}t) \qquad (4.17)$$

where $\Omega(z_R,z_I;x,y,0)$ is given by (4.14) using the initial condition $q(x,y,0)$ and $A_{30} = \mathrm{diag}(i,-i)$.

5. INVERSE PROBLEMS IN 0+1

The Lax pair associated with the PIV equation

$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2+\alpha)y + \frac{\beta}{y}, \tag{5.1}$$

is given by

$$Y_z(z) = \left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}z + \begin{pmatrix}t & u\\ \frac{2}{u}(v-\theta_0-\theta_\infty) & -t\end{pmatrix} + \begin{pmatrix}\theta_0 - v & -\frac{uy}{2}\\ \frac{2v}{uy}(v-2\theta_0) & -(\theta_0-v)\end{pmatrix}\frac{1}{z}\right]Y(z), \tag{5.2a}$$

$$Y_t(z) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}z + \begin{pmatrix}0 & u\\ \frac{2}{u}(v-\theta_0-\theta_\infty) & 0\end{pmatrix}Y(z). \tag{5.2b}$$

Indeed $Y_{zt} = Y_{tz}$ implies

$$\begin{aligned}\frac{dy}{dt} &= -4v + y^2 + 2ty + 4\theta_0, \quad \frac{du}{dt} = -u(y+2t),\\ \frac{dv}{dt} &= -\frac{2}{y}v^2 + \frac{4\theta_0}{y} - y)v + (\theta_0 + \theta_\infty)y,\end{aligned} \tag{5.3}$$

where,

$$\alpha = 2\theta_\infty - 1, \quad \beta = -8\theta_0^2.$$

As in the cases of 1+1 and 2+1, solving the initial value problem of PIV reduces to solving an inverse problem for Y: **Reconstruct $Y(z;t)$ in terms of appropriate monodromy data**. Again this inverse problem will be solved in terms of a RH problem. Thus it is essential to study the analytic properties of Y with respect to z. However, in contrast to the analogous problem in IST for 1+1 and 2+1, this task here is straightforward: Equation (5.2a) is a linear ODE in z, therefore its analytic structure is completely determined by its singular points. In this particular case $z = 0$ is a regular singular point and $z = \infty$ is an irregular singular point of rank 2. Complete information about $z = \infty$ is provided by the monodromy matrix M_∞ and by the Stokes multipliers a, b, c, d. Solutions of (5.2a), Y_0 and Y_1, normalized at zero and

infinity respectively are related via a connection matrix E_0 with entries α_0, β_0, γ_0, δ_0. Taking into consideration the above singularities, there exists a sectionally holomorphic function Y, with jumps across the four rays, arg $z = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and with singularities at $z = 0$, $z = \infty$. The jumps are specified by a, b, c, d and the nature of singularities by M_0, M_∞. This leads to a matrix, singular, discontinuous RH problem, defined on the above rays and specified in terms of the monodromy data

$$\text{Monodromy Data (MD)} = \{a,b,c,d,\ \alpha_0,\beta_0,\gamma_0,\delta_0\}.$$

A consistency condition of the above RH problem yields

$$\left(\prod_{j=1}^{4} G_j\right) M_\infty = E_0^{-1} M_0^{-1} E_0,$$

where G_j are the Stokes matrices uniquely defined in terms of the Stokes multipliers. Using (5.5) and certain similarity arguments it can be shown that all MD can be expressed in terms of two of them. Furthermore, equation (5.2b) implies that the MD are time invariant. Hence the above basic RH is specified in terms of two initial parameters (these two initial parameters are obtained from the two initial data of PIV). The solution of this RH problem yields $Y(z;t)$ and hence (5.2a) yields $y(t)$.

The above basic RH problem can be simplified considerably: (i) Assume $0 \le \theta_0 < 1$, $0 \le \theta_\infty < 1$, $\theta_0 \neq \frac{1}{2}$; then the above RH problem is regular. It is interesting that the basic RH problem can be used to obtain Schlessinger transformations which shift θ_0 and θ_∞ by a half-integer. By using these transformations the general case is reduced to the regular case. (ii) The basic RH problem can be mapped to a sequence of two RH problems, one on the line arg $z = \frac{\pi}{4}$ and the other on the line arg $z = -\frac{\pi}{4}$. The first one is continuous (both at $x = 0$ and $x = \infty$); furthermore, it can be solved in closed form. The second one is discontinuous both at $x = 0$ and $x = \infty$. By using standard auxiliary functions one maps the discontinuous problem to a continuous one. Then the theory of continuous RH problems on simple contours can be used to establish uniqueness and existence of solutions. Elementary solutions of PIV, expressible in terms of Weber-Hermite functions are natually obtained within the above formalism. We summarize the results of [7] concerning PIV.

Proposition 5.1 (Direct Problem). Let Y_0 be the solution of (5.2a) analytic in the neighbourhood of $z = 0$ and normalized by the requirements that $\det Y_0 = 1$ and that Y_0 also solves (5.2b). Let Y_j, $j = 1,\dots,4$ be solutions of (5.2a) analytic in the neighbourhood of infinity such that $\det Y_j = 1$ and $Y_j \sim Y_\infty$ as $|x| \to \infty$ in S_j, where $\sim$ denotes asymptotics, Y_∞ is the formal solution matrix of (5.2a) in the neighbourhood of infinity, and the sectors S_j are given by

$$S_1:\ -\frac{\pi}{4} \le \arg z < \frac{\pi}{4},\quad S_2:\ \frac{\pi}{4} \le \arg z < \frac{3\pi}{4},$$

$$S_3:\ \frac{3\pi}{4} \le \arg z < \frac{5\pi}{4},\quad S_4:\ \frac{5\pi}{4} \le \arg z < \frac{7\pi}{4}.$$

The rays $C_1,\dots,C_4$ are defined by $\arg z = \frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ respectively.

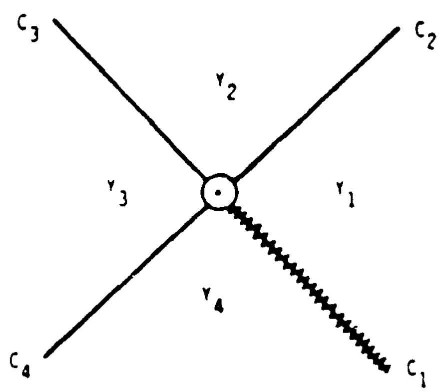


Fig. 5.1

Then the anlytic functions $Y_0, Y_1,\dots,Y_4$ satisfy:

(i) $Y_0(z) \sim \hat{Y}_0(z) z^{D_0}$ as $z \to 0$; $D_0 \neq \mathrm{Diag}(\theta_0, -\theta_0)$, $\theta_0 \neq \frac{n}{2}$, $n \in \mathbb{Z}$,

where $\hat{Y}_0(z)$ is holomorphic at $z = 0$. (If $\theta_0 = n/2$, $Y_0(z)$ has a logarithmic singularity.)

(ii) $Y_j(z) \sim \hat{Y}_\infty(z) e^{Q(z)} (1/z)^{D_\infty}$ as $|z| \to \infty$, z in S_j, $D_\infty \neq \mathrm{Diag}(\theta_\infty, -\theta_\infty)$,

$Q(z) \doteq \mathrm{Diag}(q,-q)$, $q(z,t) \doteq \frac{z^2}{2} + zt$, $\hat{Y}_\infty(z)$ is holomorphic at $z = \infty$.

(iii) $Y_0(ze^{2i\pi}) = Y_0(z)M_0$, $M_0 \doteq \begin{pmatrix} \exp(2i\pi\theta_0) & 2i\pi J\exp(2i\pi\theta_0) \\ 0 & \exp(-2i\pi\theta_0) \end{pmatrix}$, (5.4)

$J = 0$ if $\theta_0 \neq \frac{n}{2}$, $J = 1$ if $\theta_0 = \frac{n}{2}$.

(iv) $Y_2(z) = Y_1(z)G_1$, $Y_3(z) = Y_2(z)G_2$, $Y_4(z) = Y_3(z)G_3$,

$$Y_1(z) = Y_4(ze^{2i\pi})G_4M_\infty, \tag{5.5}$$

where

$$G_1 \doteq \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad G_2 \doteq \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad G_3 \doteq \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

$$G_4 \doteq \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad M_\infty \doteq \exp(2i\pi D). \tag{5.6}$$

(v) $Y_1(z) = Y_0(z)E_0$, $E_0 \doteq \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix}$, $\det E_0 = 1$. (5.7)

Furthermore, the parameters

$$MD \doteq \{a,b,c,d,\alpha_0,\beta_0,\gamma_0,\delta_0\} \tag{5.8}$$

satisfy the following consistency condition.

(vi) $\left(\prod_{j=1}^{4} G_j\right)M_\infty = E_0^{-1}M_0^{-1}E_0.$ (5.9)

Proposition 5.2 (Properties of Monodromy Data)

(i) The monodromy data, MD, given by (5.8) and defined in Proposition 5.1, are time-invriant.

(ii) All of the MD can be expressed in terms of two of them.

(iii) $(1+bc)\exp(2i\pi\theta_\infty) + [ad + (1+cd)(1+ab)]\exp(-2i\pi\theta_\infty) = 2\cos 2\pi\theta_\infty$. (5.10)

In what follows we formulate a RH problem for the case that $0 \le \theta_0 < 1$, $0 < \theta_\infty < 1$. This assumption leads to a regular RH problem. The general case follows by considering this result and Schlessinger transformations.

Theorem 5.1 (Inverse Problem). Consider the following matrix, regular homogeneous RH problem along the four rays $C_1,\ldots,C_4$ (Figure 5.1): Determine the sectionally holomorphic function $\Psi(z)$, $\Psi(z) = \Psi_j(z)$ if z is in S_j, $j = 1,\ldots,4$, from the following conditions:

1. Ψ_j satisfy the jump conditions

$$\Psi_2(\zeta) = \Psi_1(\zeta)g_1(\zeta),\ \Psi_3(\zeta) = \Psi_2(\zeta)g_2(\zeta),\ \Psi_4(\zeta) = \Psi_3(\zeta)g_3(\zeta),$$

$$\Psi_1(\zeta) = \Psi_4(\zeta\, e^{2i\pi})g_4(\zeta) \tag{5.11}$$
along the rays C_2, C_3, C_4, C_1 respectively, where

$$g_j \doteq e^{Q}G_j e^{-Q},\quad j = 1,2,3,\quad g_4 \doteq e^{Q}G_4 e^{-Q}M_\infty. \tag{5.12}$$

2. $$\Psi(z) \sim \left(\frac{1}{z}\right)^{D_\infty}\left(I + O\left(\frac{1}{z}\right)\right) \text{ as } |z| \to \infty. \tag{5.13}$$

3. $\Psi(z)$ has at most an integrable singularity at the origin with a monodromy matrix given by

$$\Psi_1(ze^{2i\pi}) = \Psi_1(z)E_0^{-1}M_0E_0,\quad z \quad 0. \tag{5.14}$$

In the above, G_j, Q, M_∞, D_∞, M_0 are defined in Proposition 5.1.

4. The monodromy data MD, given by (5.8), satisfy the properties given in Proposition 5.2(ii). Then:

(i) The above RH problem is discontinuous both at the origin and at infinity. Actually

$$\prod_{j=1}^{4} g_j \sim E_0^{-1}M_0^{-1}E_0,\ z \to 0;\qquad \prod_{j=1}^{4} g_j \sim M_\infty,\quad z \to \infty. \tag{5.15}$$

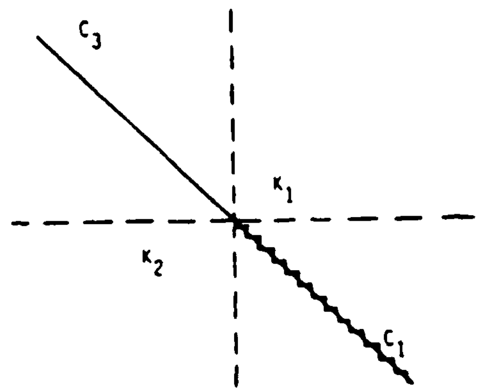


Fig. 5.2

(ii) To obtain the solution of the above RH problem consider the following RH problem along the contour $C_1 + C_3$: Determine the sectionally holomorphic function $K(z)$, $K(z) = K_1(z)$ if z in $S_1 + S_2$, $K(z) = K_2(z)$ if z in $S_3 + S_4$, from the following conditions:

1. K_j satisfy the jump condition

$$K_1 = K_2 \begin{cases} h \begin{pmatrix} 1 & de^{2q} \\ 0 & -a/c \end{pmatrix} M_\infty h^{-1} \text{ on } C_1, \\ \\ h \begin{pmatrix} 1 & -be^{2q} \\ 0 & -a/c \end{pmatrix} h^{-1} \text{ on } C_3 \end{cases} \qquad h(z) \equiv \begin{pmatrix} 1 & 0 \\ a\rho(z) & 1 \end{pmatrix},$$

$$\rho(z) \equiv -\frac{1}{2\pi i} \int_{C_2+C_4} \frac{d\zeta\, e^{-2q(\zeta)}}{\zeta - z}. \qquad (5.16)$$

(If h_1, h_2 denote h in $S_2 + S_3$ and $S_4 + S_1$ respectively then $h = h_1$ on C_1, $h = h_2$ on C_3.)

2. $K(z) \sim (\frac{1}{z})^{D_\infty}(I + O(\frac{1}{z}))$ as $|z| \to \infty$. (5.17)

3. $K(z)$ has at most an integrable singularity at the origin with a monodromy matrix given by

$$K(ze^{2i\pi}) = K(z)h_1(0)E_0^{-1}M_0E_0h_1^{-1}(0), \quad z \to 0. \tag{5.18}$$

The above RH is discontinuous both at the origin and at infinity. Actually if g_{K_1}, g_{K_3} denote the jump matrices along C_1, C_3 respectively then

$$g_{K_3}^{-1}g_{K_1} \sim h_1(0)E_0^{-1}M_0^{-1}E_0h_1^{-1}(0), \; z \to 0; \quad g_{K_3}^{-1}g_{K_1} \sim M_\infty, \; z \to \infty. \tag{5.19}$$

However, the above RH problem can be mapped to a continuous one using the auxiliary functions

$$\left(\frac{z}{z\pm 1}\right)^{\pm\theta_0}, \quad \left(\frac{1}{z\pm 1}\right)^{\pm\theta_\infty}, \tag{5.20}$$

to remove the above singularities.

Ψ is related to K via:

$$\Psi = Kh \text{ if } z \text{ in } S_1+S_2; \quad \Psi = KhM, \quad M \doteq \mathrm{Diag}(1,-a/c), \tag{5.21}$$

if z in S_3+S_4

$(i.e., \quad \Psi_1 = K_1h_1, \quad \Psi_2 = K_1h_2, \quad \Psi_3 = K_2h_1M, \; \Psi_4 = K_2h_1M).$

Proposition 5.3 (The Solution of PIV). Let $\Psi(z)$ be the solution matrix of the inverse problem formulated in Theorem 5.1. Then $y(t)$,

$$y(t) = -\left(\frac{1}{u}\frac{du}{dt} + 2t\right), \quad u \doteq -2\lim_{|z|\to\infty} \Psi_{21}(z)e^{-2q(z)}, \tag{5.22}$$

solves PIV.

6. INVERSE PROBLEMS IN n SPATIAL DIMENSIONS, n > 2

Consider the inverse problem associated with the following system of N first-order equations in n+1 dimensions:

$$\Psi_{x_0} + \sigma\sum_{\ell=1}^{n} J_\ell\Psi_{x_\ell} = q\Psi, \; \sigma = \sigma_R + i\sigma_I, \quad \sigma_I \neq 0, \; n > 1, \tag{6.1}$$

where $q(x_0,x)$ is an $N \times N$ matrix-valued off-diagonal function in $\mathbb{R}^{n+1}$, decaying suitably fast for large x_0, x, and the J_ℓ are constant real diagonal $N \times N$ matrices (we denote the diagonal entries of J_ℓ by $J_\ell^1, \ldots, J_\ell^N$). Alternatively, using the transformation

$$\Psi(z,x_0,x) = \mu(z,x_0,x)\exp[i \sum_{\ell=1}^{n} z_\ell(x_\ell - \sigma x_0 J_\ell)], \quad z \in \mathbb{C}^n, \tag{6.2}$$

equation (2.13) becomes

$$\mu_{x_0} + \sigma \sum_{\ell=1}^{n} (J_\ell \mu_x + iz_\ell[J_\ell,\mu]) = q\mu. \tag{6.3}$$

We assume that $n \leq N$, otherwise the entries of the J_ℓ matrices will be linearly related and one can always reduce n by a change of coordinates. An inverse problem in this case is defined as follows: Given appropriate inverse data T, where T is an $N \times N$ matrix-valued off-diagonal function of suitable inverse parameters, reconstruct the potential q. Again there exists a μ which is bounded for all complex z, $z \in \mathbb{C}^n$. $\partial\mu/\partial\bar{z}$ depends on appropriate inverse data $T(z_R,z_1,m_2,\ldots,m_n)$, $z_R \in \mathbb{R}^n$, $z_I \in \mathbb{R}^n$, $m_\ell \in R$. T satisfies $\frac{\partial T}{\partial\bar{z}_i\,\partial\bar{z}_j} = \frac{\partial T}{\partial\bar{z}_j\,\partial\bar{z}_i}$. Using this equation and introducing Born variables,

$$z,\ m \Leftrightarrow \quad w_0,w,\chi\ ; \quad w_0 \in \mathbb{R}, \quad w \in \mathbb{R}^n, \quad \chi \in \mathbb{C}^{n-1}, \tag{6.4}$$

one obtains a characterization equation for the inverse data:

$$\hat{T}^{ij}(w_0,w) = T^{ij}(w_0,w,\chi) - \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{d\chi'_{p_R}\,d\chi'_{p_I}\,N^{ij}_{IP}[T](w_0,w,\chi^{p'})}{\chi_p - \chi'_p}, \tag{6.5}$$

where N is a quadratic function of T. That is, $T^{ij}(z,m)$ is appropriate inverse data iff the right-hand side of (6.5) is independent of χ. Hence, equation (6.5) serves as both characterizing T^{ij} and defining $\hat{T}^{ij}$. This equation was first introduced by Nachman and Ablowitz [8]. Using equation (6.5) and taking the limit of μ as $|\chi| \to \infty$ we show that the general problem of reconstructing an $N \times N$ potential q in n+1 spatial dimensions, is reduced to one of reconstructing a 2×2 potential with entries q^{ij}, q^{ji} in two dimensions. The inverse data needed for this reconstruction is precisely $\hat{T}^{ij}, \hat{T}^{ji}$. This reduction makes crucial use of the existence of redundant scattering parameters. In this sense it is the analog of the Born approximation. However, the crucial difference

is that while in the inverse scattering of the multidimensional Schrödinger equation one can reconstruct the potential in closed form, here one can only reduce the general problem to one for 2×2 matrices in two dimensions. This reduced problem was solved in [6]. In the following, we summarize the results of [9].

Proposition 6.1 (Bounded Eigenfunctions). The function $\mu(x_0,x,z)$ defined below, solves equation (6.3), is bounded for all complex values of z and tends to I for large k:

$$\mu^{ij}(x_0,x,z) = \delta^{ij} + \frac{\mathrm{sgn}(\sigma_I J_1^i)}{2\pi i} \int_{\mathbb{R}^2} d\xi_0 d\xi_1 \frac{\exp[i\beta^{ij}(x_0-\xi_0,x_1-\xi_1,z)]}{(x_1-\xi_1) - \sigma J_1^i(x_0-\xi_0)}$$

$$(q\mu)^{ij}(\xi_0,\xi_1,x_2-(x_1-\xi_1)J_2^i/J_1^i,\ldots,x_n-(x_1-\xi_1)J_n^i/J_1^i,z), z \in \mathbb{C}^n, \quad (6.5)$$

where β^{ij} is defined by

$$\beta^{ij}(x_0,x_1,z) = \sum_{\ell=1}^{n} \frac{J_\ell^i - J_\ell^j}{\sigma_I} \left[x_0|\sigma|^2 z_{\ell_I} - \frac{x_1(\sigma z_\ell)_I}{J_1^i} \right. , z_\ell = z_{\ell_R} + i z_{\ell_I}. \quad (6.6)$$

Equivalently μ_{ij} satisfies

$$\mu^{ij}(x_0,x,z) = \delta^{ij} + \frac{\mathrm{sgn}(\sigma_I J_1^i)}{2\pi i} \int_{\mathbb{R}^{n+1}} d\xi_0 d\xi [c_{n-1}\int_{\mathbb{R}^{n-1}} dm^2 e^{i\alpha^i(x-\xi,m)}]$$

$$\frac{\exp[i\beta^{ij}(x_0-\xi_0,x_1-\xi_1,z)](q\mu)^{ij}(\xi_0,\xi,z)}{x_1-\xi_1-\sigma J_1^i(x_0-\xi_0)}, \quad (6.7)$$

where

$$dm^2 = dm_2\ldots dm_n,\ \alpha^i(x,m) = \sum_{\ell=2}^{n} m_\ell(x_\ell - x_1 \frac{J_\ell^i}{J_1^i}),\ c_n = \frac{1}{(2\pi)^n}. \quad (6.8)$$

Proposition 6.2 (Departure from Holomorphicity). Let μ^{ij} be defined by eq. (6.5). Then

$$\frac{\partial\mu}{\partial\bar{z}_p}(x_0,x,z) = \sum_{i,j}\gamma^i(J_p^i - J_p^j)\exp[i\beta^{ij}(x_0,x_1,z)]$$

$$\times\ c_{n-1}\int_{\mathbb{R}^{n-1}} dm^2\exp[i\alpha^i(x,m)]T^{ij}(z,m)\mu(x_0,x,\lambda^{ij}(z,m))E_{ij}, \qquad (6.9)$$

where $\beta^{ij}(x_0,x_1,z)$, $\alpha^i(x,m)$ are defined by (6.6), 6.8) respectively; E_{ij} is an $N \times N$ matrix with zeros in all its entries except the ij^{th}, which equals 1; and λ^{ij} and T^{ij} are given by

$$\lambda_1^{ij}(z,m) \doteq (z_{1_R}^{ij} - \sum_{\ell=2}^{n} m_\ell \frac{J_\ell^i}{J_1^i}, z_{1_I}), \quad \lambda_r^{ij}(z,m) = (z_{r_R}+m_r, z_{r_I}); \ r=2,\ldots,n.$$

$$\gamma^i \doteq \bar{\sigma}/4\pi i|J_1^i\sigma_I|,$$

$$T^{ij}(z,m) \doteq \int_{\mathbb{R}^{n+1}} d\xi_0 d\xi\ \exp[-i\beta^{ij}(\xi_0,\xi_1,z)-i\alpha^i(\xi,m)](q\mu)^{ij}(\xi_0,\xi,z). \qquad (6.10)$$

Proposition 6.3 (Characterization of T)

(a) Assume that $\partial\mu/\partial\bar{z}_p$ is given by Eq. (6.9) and the $T^{ij}(z,m)$ is given by (6.10). Then

$$L_{rp}^{ij}T^{ij}(z,m) = -\sum_{\ell=1}^{n} c_{n-1}\int_{\mathbb{R}^{n-1}} dM^2 T^{i\ell}(\lambda^{\ell j}(z,M),m-M)T^{\ell j}(z,m)$$

$$\times[(J_p^\ell - J_p^j)(J_r^i - J_r^\ell) - (J_r^\ell - J_r^j)(J_p^i - J_p^\ell)] \doteq N_{rp}^{ij}[T](z,m), \qquad (6.11)$$

where

$$L_{rp}^{ij} \doteq (J_p^i - J_p^j)\frac{\partial}{\partial\bar{z}_r} - (J_r^i - J_r^j)\frac{\partial}{\partial\bar{z}_p}. \qquad (6.12)$$

(b) Assume that $\partial\mu/\partial\bar{z}_p$ is given by Eq. (6.9) and that $\partial^2\mu/\partial\bar{z}_r\partial\bar{z}_p$ is symmetric with respect to r, p. Then $T^{ij}(z,m)$ solves (6.11).

Following A. Nachman and M. J. Ablowitz we introduce appropriate **Born variables**. Then equation (6.11) can be integrated. Furthermore, we can compute the limit of T^{ij} in the new coordinates as $|\chi_p| \to \infty$ (see below):

Let w_0^{ij}, w_1^{ij}, w_ℓ, $\ell = 2,\ldots,n \in \mathbb{R}^1$ and $\chi_\ell \in \mathbb{C}^1$, $\ell = 2,\ldots,n$, be defined by

$$w_0^{ij} \equiv \sum_{r=1}^{n} \frac{J_r^i - J_r^j}{\sigma_I} |\sigma|^2 z_{r_I}, \quad w_1^{ij} \equiv - \sum_{r=1}^{n} \frac{J_r^i - J_r^j}{\sigma_I J_1^i} (\sigma z_r)_I - \sum_{r=2}^{n} m_r \frac{J_r^i}{J_1^i},$$

$$w_\ell \equiv m_\ell, \qquad \chi_\ell^{ij} \equiv \frac{z_\ell}{J_1^j - J_1^i}, \; \ell = 2,\ldots,n. \tag{6.13}$$

Assume that

$$(J_1^r - J_1^j)(J_p^i - J_p^j) \neq (J_1^i - J_1^j)(J_p^r - J_p^j), \text{ for all distinct } i,j,r \text{ and } p \neq 1. \tag{6.14}$$

For convenience of writing we usually suppress the superscripts, i,j in w_0, w_1, χ. Let z denote $z_1,\ldots,z_n$, m denote $m_2,\ldots,m_n$, χ denote $\chi_2,\ldots,\chi_n$, w denote $w_1,\ldots,w_n$. Then we have the following.

(a) The inverse of the transformation $z, m \to w_0, w, \chi$ is given by

$$z_\ell = \chi_\ell (J_1^j - J_1^i), \quad m_\ell = w_\ell, \; \ell = 2,\ldots,n, \quad z_1 = - \sum_{r=2}^{n} (J_r^j - J_r^i)\chi_r + \frac{(\bar{\sigma}/|\sigma|^2) w_0 + \sum_{r=1}^{n} w_r J_r^i}{J_1^j - J_1^i}. \tag{6.15}$$

(b) In the new coordinates, Eq. (6.11) with $r = 1$ becomes

$$\frac{\partial T^{ij}}{\partial \bar{\chi}_p}(w_0, w, \chi) = N_{1p}^{ij}[T](w_0, w, \chi), \; p = 2,\ldots,n. \tag{6.16}$$

(c) In the new coordinates,

$$T^{ij}(w_0, w, \chi) = \int_{\mathbb{R}^{n+1}} d\xi_0 d\xi \exp[-i(w_0\xi_0 + w\xi)](qu)^{ij}(\xi_0, \xi, w_0, w, \chi),$$

where

$$w\xi = \sum_{r=1}^{n} w_r \xi_r. \tag{6.17}$$

(d) Let

$$\mu_i^{\ell j} \triangleq \mu^{\ell j}(x_0, x, w_0^{ij}, w^{ij}, \chi^{ij}), \quad \hat{\mu}_i^{\ell j} = \lim_{|\chi_p| \to \infty} \mu_i^{\ell j}.$$

Then the $\hat{\mu}_i^{\ell j}$ satisfy

$$\hat{\mu}_i^{ij}(x_0, x, w_0, w) = \frac{\operatorname{sgn}(\sigma_I J_1^i)}{2\pi i} c_{n-1} \int_{\mathbb{R}^{2n}} \times$$

$$\frac{dx_0' dx' dw^2 \exp[i((x_0 - x_0')w_0 \quad + (x - x')w)]}{x_1 - x_1' - \sigma J_1^i(x_0 - x_0')} q^{ij}(x_0', x') \hat{\mu}_i^{ij}(x_0', x', w_0, w),$$

$$\hat{\mu}_i^{jj}(x_0, x, w_0, w) = 1 + \frac{\operatorname{sgn}(\sigma_I J_1^j)}{2\pi i} c_{n-1} \int_{\mathbb{R}^{2n}} \times$$

$$\frac{dx_0' dx' dW^2 q^{ji}(x_0', x') \hat{\mu}_i^{ij}(x_0', x', w_0, w)}{x_1 - x_1' - \sigma J_1^j(x_0 - x_0')}, \; \hat{\mu}_i^{\ell j} = 0, \text{ for all } \ell, \ \ell \neq i, \ell \neq j. \tag{6.19}$$

(e) $$\lim_{|\chi_p| \to \infty} T^{ij}(w_0, w, \chi) = \int_{\mathbb{R}^{n+1}} d\xi_0 d\xi \exp[-i(w_0 \xi_0 + w\xi)] \times$$

$$q^{ij}(\xi_0, \xi) \hat{\mu}_i^{ij}(\xi_0, \xi, w_0, w) \triangleq \hat{T}^{ij}(w_0, w). \tag{6.20}$$

(f) The basic characterization equation is given by

$$\hat{T}^{ij}(w_0, w) = T^{ij}(w_0, w, \chi) - \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{d\chi_{p_R}' d\chi_{p_I}' N_{1p}^{ij}[T](w_0, w, \chi^{p'})}{\chi_p - \chi_p'}. \tag{6.21}$$

where $\chi^{p'}$ denotes $\chi_2, \ldots, \chi_{p-1}, \chi_p', \chi_{p+1}, \ldots, \chi_n$.

It follows from the above that as $|\chi_p| \to \infty$, the μ^{ij}'s decouple. Furthermore, the $\hat{\mu}_i^{ij}$, $\hat{\mu}_i^{jj}$ satisfy a system of two equations depending on q^{ij}, q^{ji}. It turns out that: (a) By introducing appropriate spatial variables ξ, the $\hat{\mu}_i^{ij}$, $\hat{\mu}_i^{jj}$ satisfy equations in two spatial dimensions.

(b) The inverse data needed to reconstruct $\hat{\mu}_i^{ij}$, μ_i^{jj} (and hence q^{ij}, q^{ji}) can be obtained from $\hat{T}^{ij}$:

Proposition 6.4 (Reconstruction of q). Let

$$\alpha_r \doteq \frac{J_2^j J_r^i - J_r^j J_2^i}{J_1^i J_2^j - J_1^j J_2^i}, \quad \beta_r = \frac{J_1^i J_r^j - J_1^j J_r^i}{J_1^i J_2^j - J_1^j J_2^i}, \quad r = 1,\dots,n, \tag{6.22}$$

where for convenience of writing we have suppressed the dependence of α_r, β_r on i, j. Let $\xi_0 \in \mathbb{R}$, $\xi \in \mathbb{R}^n$,

$$x_0 = \xi_0, \quad x_1 = \xi_1, \quad x_2 = \xi_2, \tag{6.23}$$

$$x_\ell = \xi_\ell + \alpha_\ell \xi_1 + \beta_\ell \xi_2, \quad \ell = 3,\dots,n.$$

Then we have the following:

(a) The system (6.19) becomes

$$\hat{\mu}_i^{ij}(\xi_0,\xi,\hat{z}) = \operatorname{sgn} \frac{\sigma_I J_1^i}{2\pi i} \int_{\mathbb{R}^2} d\xi_0' d\xi_1' [\xi_1 - \xi_1' - \sigma J_1^i(\xi_0 - \xi_0')]^{-1}$$

$$\times \exp[i\hat{\beta}^{ij}(\xi_0-\xi_0', \xi_1-\xi_1', \hat{z})] q^{ij} \hat{\mu}_i^{jj}(\xi_0', \xi_1', \xi_2 - (\xi_1-\xi_1')\frac{J_2^i}{J_1^i}, \xi_3,\dots,\xi_n, z),$$

$$\hat{\mu}_i^{jj}(\xi_0,\xi,\hat{z}) = 1 + \operatorname{sgn} \frac{\sigma_I J_1^j}{2\pi i} \int_{\mathbb{R}^2} d\xi_0' d\xi_1' [\xi_1 - \xi_1' - \sigma J_1^j(\xi_0 - \xi_0')]^{-1}$$

$$\times q^{ji} \hat{\mu}_i^{ij}(\xi_0', \xi_1', \xi_2 - (\xi_1-\xi_1')\frac{J_2^j}{J_1^j}, \xi_3,\dots,\xi_n, \hat{z}), \tag{6.24}$$

where

$$\hat{z} \doteq \sum_{r=1}^{n} (z_r \alpha_r + \frac{J_2^j - J_1^j}{J_1^j - J_1^i} z_r \beta_r), \quad \hat{\beta}^{ij}(x_0, x_1, \hat{z})$$

$$\hat{\beta}^{ij}(x_0,x_1,\hat{z}) \doteq \frac{J_1^i - J_1^j}{\sigma_I} [x_0 |\sigma|^2 z_I - x_1 \frac{(\sigma z)_I}{J_1^i}]. \tag{6.25}$$

(b) $\hat{T}^{ij}$ in the new coordinates becomes

$$\hat{T}^{ij}(\hat{z},\hat{m}) = \int_{\mathbb{R}^{n+1}} d\xi_0' d\xi' \exp[-i\hat{\beta}^{ij}(\xi_0', \xi_1', \hat{z}) +$$

$$+ \hat{m}_2(\xi_2' - \xi_1' \frac{J_2^i}{J_1^i}) + \sum_{r=3}^{n} \hat{m}_r\xi_r']q^{ij}\hat{\mu}_i^{jj}(\xi_0',\xi',\hat{z}), \tag{6.26}$$

where

$$\hat{m}_2 \doteq m_2 + \sum_{r=3}^{n} m_r\beta_r, \ \hat{m}_\ell = m_\ell, \quad \ell = 3,\ldots,n. \tag{6.27}$$

(c) The inverse data associated with (6.24) and the analogous problem for $\hat{\mu}_j^{ji}$, $\hat{\mu}_j^{ii}$ are given by T^{ij}, T^{ji}. Let

$$T^{ij}(z,\xi_2-\xi_1 \frac{J_2^i}{J_1^i}, \xi_3,\ldots,\xi_n) \doteq c_{n-1} \int_{\mathbb{R}^{n-1}} d\hat{m} \exp[i\hat{m}_2(\xi_2-\xi_1 \frac{J_2^i}{J_1^i})$$

$$+ i \sum_{r=3}^{n} \hat{m}_r\xi_r]\hat{T}^{ij}(\hat{z},\hat{m}). \tag{6.28}$$

Then

$$\dot{T}^{ij}(\hat{z},\xi_2-\xi_1 \frac{J_2^i}{J_1^i},\xi_3,\ldots,\xi_n) \doteq \int_{\mathbb{R}^2} d\xi_0'd\xi_1' \exp[-i\hat{\beta}^{ij}(\xi_0',\xi_1',z)]$$

$$\times(q^{ij}\hat{\mu}_i^{jj})(\xi_0',\xi_1',\xi_2 - (\xi_1-\xi_1') \frac{J_2^i}{J_1^i}, \xi_3,\ldots,\xi_n,\hat{z}). \tag{6.29}$$

Equations (6.1)-(6.3) with $\sigma = -1$ lead to a system which appears to be physically more interesting: (a) Since the system is hyperbolic one may consider the physically important question of inverse scattering (IS); i.e., given a scattering amplitude function $S(\lambda,k)$ find the potential $q(x_0,x)$. (b) A special case of the above system, namely if the J_ℓ's are constrained by

$$\frac{J_p^\ell - J_p^j}{J_r^\ell - J_r^j} = \frac{J_p^i - J_p^j}{J_r^i - J_r^j}, \quad p, r = 1,\ldots,n, \quad i,j,\ell = 1,\ldots,N, \tag{6.6}$$

is associated with the N-wave interaction in n+1 spatial and one temporal dimensions [10]. The above system can be considered as a limiting case of (6.1)-(6.3) [8]. Alternatively, it can be considered on its own right [11]; the problem of reconstruction can be reduced to one for a 2 × 2 matrix problem in two spatial dimensions.

ACKNOWLEDGEMENTS

This work was partially supported by the Office of Naval Research under

Grant Number N00014-76-C-0867 and the National Science Foundation under Grant Number DMS-8501325. A. S. Fokas thanks Professor Lakshmanan and his school for their warm hospitality.

REFERENCES

[1] R. G. Newton, **Scattering Theory of Waves and Particles** (Springer, Berlin, 1982).
[2] F. D. Gakhov, **Boundary Value Problems** (Pergamon, 1966); N. P. Vekua, **Systems of Singular Integral Equations** (Gordon and Breach, 1967).
[3] S. Prossdorf, **Some Classes of Singular Equations** (North-Holland, 1978).
[4] L. Hörmander, **Complex Analysis in Several Variables** (Van Nostrand, 1966).
[5] R. Beals and R. R. Coifman, Comm. Pure Appl. Math. 37 (1984) 39.
[6] A. S. Fokas, Phys. Rev. Lett. 51 (1983); A. S. Fokas and M. J. Ablowitz, J. Math. Phys. 25 (1984) 2505.
[7] A. S. Fokas, U. Mugan, M. J. Ablowitz, A Method of Solution for Painlevé Equations: PIV, PV, preprint INS#73 (1987).
[8] A. I. Nachman, M. J. Ablowitz, Stud. Appl. Math. 71 (1984) 251.
[9] A. S. Fokas, J. Math. Phys. 27 (1986) 1737.
[10] M. J. Ablowitz and R. Haberman, Phys. Rev. Lett. 35 (1975) 1125.
[11] A. S. Fokas, Phys. Rev. Lett. 57 (1986) 159.
[12] A. S. Fokas and M. J. Ablowitz, Comm. Math. Phys. 91 (1983) 381.
[13] A.V. Mikhailov, Physica 3D (1981) 1&2, 73-117

END
DATE
FILMED
DTIC
JULY 88